LRSR Folk Theorem - Example

ECON 201B - Game Theory

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Consider the following two player simultaneous move game (We will call ROW player 1 and COL player 2)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1,2</td>
<td>1,6</td>
<td>8,5</td>
<td>2,3</td>
</tr>
<tr>
<td>D</td>
<td>1,2</td>
<td>2,3</td>
<td>9,5</td>
<td>4,6</td>
</tr>
</tbody>
</table>

Suppose the game is played only once

a) Find all Nash equilibria ($n^1$) of this game and payoffs for player 1

Best responses are denoted in bold. It is clear that the unique pure NE is $(D, S)$ such that the payoff for player 1 is 4. It’s easy to check there is no mixed NE since player 1 will randomize between U and D only if player 2 plays L for sure. But in fact player 2 will never play L since this is a strictly dominated strategy. Hence,

$$n^1 = 4$$

b) Find the minmax ($m^1$) for player 1

The idea of the minmax is to find which is the worst possible punishment player 2 can impose to player 1. To find it, we should first ask what is the best response by player 1 to each possible action player 2 can take (this is the $\text{max}$ part on the name $\text{minmax}$). This basically says that whatever the action player 2 takes, player 1 will always play the best response to it and, in

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*These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu*
In this way, there is no possibility for player 2 to just impose any punishment she wants (for example, the minimum payoff in the matrix).

After we find the best responses, we need to find the minimum of those numbers, (this is the \textit{min} part on the name \textit{minmax}).

In symbols,

\[ m^1 = \min_{a^2} \left[ \max_{a^1} u^1(a^1, a^2) \right] \]

In this game

<table>
<thead>
<tr>
<th>( \alpha^2 )</th>
<th>( a^1 \in BR^1(\alpha^2) )</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>U,D</td>
<td>1</td>
</tr>
<tr>
<td>M</td>
<td>D</td>
<td>2</td>
</tr>
<tr>
<td>R</td>
<td>D</td>
<td>9</td>
</tr>
<tr>
<td>S</td>
<td>D</td>
<td>4</td>
</tr>
</tbody>
</table>

and the minimum payoff among the possibilities is

\[ m^1 = 1 \]

\textbf{IMPORTANT:} In this case I’m not considering mixed strategies by player 2 since it’s clear the worst thing that can happen to player 1 is player 2 to play L. HOWEVER, in general (particularly when player 2 has only two possible actions) we need to check also which are the best responses by player 1 to mixing strategies by player 2 and the corresponding payoffs in order to obtain the minimum (remember, the minimum should be computed over \( \alpha^2 \), not over \( a^2 \)).

c) \textbf{Find the pure Stackelberg equilibrium (ps\textsuperscript{1}) payoff to player 1 moving first.}

The idea of Stackelberg (or precommitment) is to find which is the best possible result player 1 can obtain if he happens to move first (or, which is the same, to commit himself to play a certain action). To find it, we should ask first which is the best response by player 2 to each possible action player 1 can take. After knowing this, we need to ask ourselves in which case player
1 can get the maximum possible payoff.

In symbols,
\[ ps^1 = \max_{(a^1, \alpha^2) \mid \alpha^2 \in BR^2(a^1)} u^1(a^1, \alpha^2) \]

In this game

<table>
<thead>
<tr>
<th>(a^1)</th>
<th>(\alpha^2 \in BR^2(a^1))</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>M</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>S</td>
<td>4</td>
</tr>
</tbody>
</table>

and the maximum payoff among the possibilities is

\[ ps^1 = 4 \]

c) **Find the mixed Stackelberg equilibrium \((ms^1)\) payoff to player 1 moving first.**

The intuition and the method to obtain the mixed Stackelberg (or pre-commitment) equilibrium is identical to the ones used to obtain \(ps^1\) before. The only difference arises by allowing player 1 to precommit also to mixed strategies (or mix among the pure actions).

In symbols,
\[ ps^1 = \max_{(a^1, \alpha^2) \mid \alpha^2 \in BR^2(a^1)} \sum_{a^1} u^1(\alpha^1, \alpha^2)\alpha^1(a^1) \]

Note that allowing mixed strategies, the maximum is taken over \(\alpha^1\) and not over \(a^1\). Hence we need to think in expected payoffs rather than in certain payoffs.

In this game, a first step is to get the best responses by player 2 to each possible mixing by player 1.

Call \(p = \Pr(a^1 = U) = \alpha^1(U)\) the probability player 1 plays U.

1) L will never be played since it is strictly dominated.
2) M will be preferred to R if \(6p + 3(1 - p) > 5\) or \(p > \frac{2}{3}\)
3) M will be preferred to S if \(6p + 3(1 - p) > 3p + 6(1 - p)\) or \(p > \frac{1}{2}\)
4) R will be preferred to S if \(5 > 3p + 6(1 - p)\) or \(p > \frac{1}{3}\)
Hence,

<table>
<thead>
<tr>
<th>$\alpha^1$</th>
<th>$\alpha^2 \in BR^2(a^1)$</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 0$</td>
<td>S</td>
<td>4</td>
</tr>
<tr>
<td>$0 &lt; p &lt; \frac{1}{3}$</td>
<td>S</td>
<td>$(\frac{10}{3}, 4)$</td>
</tr>
<tr>
<td>$p = \frac{1}{3}$</td>
<td>S,R,mix</td>
<td>$[\frac{10}{3}, \frac{20}{3}]$</td>
</tr>
<tr>
<td>$\frac{1}{3} &lt; p &lt; \frac{2}{3}$</td>
<td>R</td>
<td>$(\frac{25}{3}, \frac{26}{3})$</td>
</tr>
<tr>
<td>$p = \frac{2}{3}$</td>
<td>R,M,mix</td>
<td>$[\frac{4}{3}, \frac{26}{3}]$</td>
</tr>
<tr>
<td>$\frac{2}{3} &lt; p &lt; 1$</td>
<td>M</td>
<td>$(1, \frac{4}{3})$</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>M</td>
<td>1</td>
</tr>
</tbody>
</table>

Let me discuss a couple of cases to be clear.

When $0 < p < \frac{1}{3}$, player 2 will play S for sure. At one extreme, when $p = 0$, the expected payoff is 4. At the other extreme, when $p = \frac{1}{3}$, the expected payoff is $\frac{10}{3}$. That’s why the expected payoffs are in the open interval $(\frac{10}{3}, 4)$.

When $p = \frac{1}{3}$, player 2 is indifferent between playing S or R, but the payoffs for player 1 will depend on her actions. If she decides to play S for sure, the expected payoff to player 1 is $\frac{10}{3}$. If she decides to play R for sure, the expected payoff to him is $\frac{26}{3}$. Hence, depending on the actual play by player 2 the expected payoffs to player 1 will be in the closed interval $[\frac{10}{3}, \frac{26}{3}]$.

The maximum payoff among all possibilities is

$$ms^1 = \frac{26}{3}$$

which is the case when player 1 mixes with $p = \alpha^1(U) = \frac{1}{3}$ and player 2 plays $R$.

Now suppose the game is infinitely repeated between a long-run player 1 and short run player 2.

e) For large $\delta$ find $\overline{v}^1$ the best equilibrium payoff for the long run player 1.

This is the maximum payoff a long run player can get in the dynamic equilibrium of a repeated game.
In symbols,

$\pi^1 = \max_{(\alpha^1, \alpha^2) | \alpha^2 \in BR^2(\alpha^1)} \left[ \min_{\alpha^1 | \alpha^1 > 0} u^1(\alpha^1, \alpha^2) \right]$

In plain words this means that, for each possible value $\alpha^2 \in BR^2(\alpha^1)$ we have to take the minimum payoff that can be obtained considering all the actions that can be played with positive probability. This is called support. Hence, the equation above considers actions that can be played by player 1 with a positive probability, no mixing strategies by player 1. This is why the part of the equation that takes minimum refers to the worst payoff in the support for each possible strategy (pure or mixed) played by player 2 as a best response.

In this case,

<table>
<thead>
<tr>
<th>$\alpha^1$</th>
<th>$\alpha^2 \in BR^2(\alpha^1)$</th>
<th>Support</th>
<th>Payoff</th>
<th>Worst in support</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 0$</td>
<td>S</td>
<td>D</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$0 &lt; p &lt; \frac{1}{3}$</td>
<td>S</td>
<td>U-D</td>
<td>2 - 4</td>
<td>2</td>
</tr>
<tr>
<td>$p = \frac{1}{3}$</td>
<td>S,R,mix</td>
<td>U-D</td>
<td>[2,8] - [4,9]</td>
<td>[2,8]</td>
</tr>
<tr>
<td>$\frac{1}{3} &lt; p &lt; \frac{2}{3}$</td>
<td>R</td>
<td>U-D</td>
<td>8 - 9</td>
<td>8</td>
</tr>
<tr>
<td>$p = \frac{2}{3}$</td>
<td>R,M,mix</td>
<td>U-D</td>
<td>[1,8] - [2,9]</td>
<td>[1,8]</td>
</tr>
<tr>
<td>$\frac{2}{3} &lt; p &lt; 1$</td>
<td>M</td>
<td>U-D</td>
<td>1 - 2</td>
<td>1</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>M</td>
<td>U</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Again, let’s consider some cases to understand what’s on the table.

Consider the case $0 < p < \frac{1}{3}$. The support in this case (actions that can be played with positive probability by player 1 under the mixing strategy $\alpha^1$) are U and D. The best response by player 2 ($\alpha^2$) is S. The payoff for player 1 in the support U (U,S) is 2 and in the support D, (D,S) is 4. Naturally the worst in support is 2.

Consider the case $p = \frac{1}{3}$. The support in this case is also both U and D. The best response for player 2 is anything between S and R. The payoff for player 1 in the support U depends on her mixing between S and R, specifically it can be any number in the closed interval [2,8]. In the case
of the support $D$, the payoff for player 1 can be any number in the closed interval $[4, 9]$. Then, the worst payoff in the support for a given mixing by player 2 is the interval $[2, 8]$.

The best equilibrium payoff will be given by the maximum of these numbers. Hence,

$$\bar{v}^1 = 8$$

which is the case when player 1 plays $U$ and player 2 plays $R$.

f) Find the critical value of $\delta$ for which $\bar{v}^1$ is an equilibrium for larger $\delta$.

In the previous step we assumed a high enough $\delta$ in order to get the best possible equilibrium payoff $\bar{v}^1$. In this step we’re asking specifically how big $\delta$ should be.

To answer this question let’s consider the case in which the punishment for deviating from the best equilibrium is to go to the NE payoff. This is called a Nash equilibrium threat.

As was discussed, $\bar{v}^1 = 8$ is obtained when player 1 plays in the support $U$ and player 2 plays $R$. The problem is that, in this case player 1 has an incentive to deviate by playing $D$ and getting 9 instead of 8 (flipping to the other support). In this case we need to punish him by playing the NE forever after.

Hence, the condition for player 1 to play $U$ when player 2 has decided to play $R$ instead of deviating and play $D$ is, $8 \geq (1 - \delta)9 + \delta4$, which imples that $\bar{v}^1 = 8$ is sustainable as an equilibrium when $\delta \geq \frac{1}{5}$.

g) Describe equilibrium strategies for both players that give $\bar{v}^1$.

First, let’s assume a public randomization device exists. Assume it consists of a bent coin (so the state space is $\Omega = \{H, T\}$, being $H$ heads and $T$ tails).

In order to sustain $\bar{v}^1$ we need strategies that, in some way, make the playing of $(U, R)$ and $(D, R)$ happen many times before going to the NE and that this situation represents an equilibrium (players do not want to deviate).
Consider the following strategies:

**Player 1:**

1. Start playing the mixed strategy $\frac{1}{2}U + \frac{1}{2}D$ (say this is decided by flipping a first coin)
2. If the first coin says "play $U$", go back to step i)
3. If the first coin says "play $D$", use the public randomization device (a second coin) to play $\frac{1}{2}U + \frac{1}{2}D$ again if $H$ and to play $D$ if $T$.
4. If there ever was a deviation from these strategies or if $(D,S)$ was ever played, then play $D$ forever after, otherwise go back to step i).

**Player 2:**

1. Start playing the pure strategy $R$
2. If it turns out player 1 played $U$, go back to step i)
3. If it turns out player 1 played $D$, use the public randomization device (the eventual second coin) to play $R$ again if $H$ and to play $S$ if $T$.
4. If there ever was a deviation from these strategies or if $(D,S)$ was ever played, then play $S$ forever after, otherwise go back to step i).

Following these strategies, the average present value for player 1 ($v^1$) is

\[
v^1 = \frac{1}{2} \left[ (1 - \delta)8 + \delta v^1 \right] + \frac{1}{2} \left[ (1 - \delta)9 + \delta [xv^1 + (1 - x)4] \right]
\]

where $x$ is the probability of head ($H$) in the second coin.

This expression basically means that, if the first coin decides player 1 to play $U$, and he follows the coin, the payoff will be 8 in the current period and the same continuation value in the future since the first coin is bent again. The problem is that an incentive for player 1 to play $U$ instead of $D$ should exists (otherwise he would deviate and get 9 instead of 8).

This incentive is given by introducing a punishment by playing $D$ such that player 1 is exactly indifferent between paying $U$ or $D$ after the first coin. Even when playing $D$ allows to grab 9 in the current period, the flipping of the second coin introduces a chance of going to the NE forever after (with a probability $(1 - x)$).
If we want player 1 to get $\nu^1 = \tau^1 = 8$ (by replacing above)

$$8 = \frac{1}{2} [(1 - \delta)8 + \delta8] + \frac{1}{2} [(1 - \delta)9 + \delta[8 + (1 - x)4]]$$

From here we can get the $x$ that makes 1 indifferent between playing U or D and allows to obtain exactly $\tau^1 = 8$. In this case, we need that 

$$(1 - \delta)9 + \delta[x8 + (1 - x)4] = 8$$

or

$$x = \frac{5\delta - 1}{4\delta}$$

It’s important to note that $x$ is well defined for $\delta \geq \frac{1}{5}$. This is not a coincidence but it’s a general principle developed in Fudenberg, Levine and Maskin (1989). The trick to obtain these strategies is to choose $x$ such that the dynamic payoff to player 1 is the same regardless of the pure strategy played amongst the strategies with positive probability.

Finally, it’s important to check the strategies are followed by both players. Given player 2 plays R, player 1 will not want to deviate from the first coin by construction (given $x$, he’s indifferent between playing U or D). Hence he will effectively randomizes between U or D. In case the second coin decides player 2 to play S (by being tails), player 1’s best response is D.

Given player 1 randomizes, player 2 will play R. In case the second coin decides player 1 to play D, player 2’s best response is to play S.

Hence, no player has an incentive to deviate from the strategies that given a $\tau^1 = 8$ to the first player.

h) For large $\delta$ find $\underline{\nu}^1$ the worst equilibrium payoff for the long run player 1.

This is the minimum payoff a long run player can get in the dynamic equilibrium of a repeated game

In symbols,

$$\underline{\nu}^1 = \min_{(a^1, a^2) \in BR^2(\tau^1)} \left[ \max_a u^1(a^1, \alpha^2) \right]$$
As can be seen, this is almost the same definition as the minmax, but with the restriction that the play of player 2 ($\alpha^2$) should be a best response for some strategy by player 1. Hence, this is a constrained minmax in the sense that we should compute it without considering strictly dominated strategies that would never be considered by player 2.

In our case this is the same than computing the minmax without considering the case in which player 2 play L since this is never her best response. Hence

$$\nu^1 = 2$$

which is the case when player 1 plays D and player 2 plays M.

i) **Find the critical value of $\delta$ for which $\nu^1$ is an equilibrium for larger $\delta$.**

In the previous step we assumed a high enough $\delta$ (you’re reading well, HIGH ENOUGH $\delta$ !!!!) in order to get the best possible equilibrium payoff $\nu^1$. In this step we’re asking specifically how big $\delta$ should be.

Recall that, by the minimization problem (and considering the case in which player 2 plays M),

- For action D, $\nu^1 \geq (1 - \delta) u^1(D, M) + \delta w(D)$
- For action U, $\nu^1 \geq (1 - \delta) u^1(U, M) + \delta w(U)$
- RC $\nu^1 \leq w(D), w(U) \leq n^1$

The first equation follows because it’s not possible to have $(D, M)$ being played since player 2 would never play M. This is because M is not a best response to D (recall player 2 must play a static best response since she does not care about the future).

Hence, the second condition must follow with equality. Considering the third equation,

$$\nu^1 \leq (1 - \delta) u^1(U, M) + \delta n^1$$

Hence,

$$2 \leq (1 - \delta)1 + \delta 4$$

which means $\nu^1$ can be achieved only when $\delta \geq \frac{1}{3}$
The intuition is that, for it to be possible that player 1 gets a lifetime utility of 2, he must play U while she must play M for some time before moving to the NE. In other words, player 1 should suffer some periods by getting only 1 if he wants to flip to gain 4 in the NE. It’s in this sense that player 1 should be patient enough to be able to play an action that gives him a low payoff if he wants a possibility in the future to move to a better situation.

j) Describe equilibrium strategies for both players that give $v^1$.

Assume a public randomization device exists. Assume it consists of a bent coin (so the state space is $\Omega = \{H, T\}$, being $H$ heads and $T$ tails).

In order to sustain $v^1$ we need strategies that, in some way, assure that in equilibrium the game occurs most of the times in (U,M) and then moves to the NE. Consider the following strategies:

**Player 1:**

i) Start playing the pure strategy $U$

ii) If there were no deviations by any player so far, then use the public randomization device to go back to step i) if $H$ and play $D$ forever if $T$.

iii) If there ever was a deviation from these strategies go back to step i).

**Player 2:**

i) Start playing the pure strategy $M$

ii) If there were no deviations by any player so far, then use the public randomization device to go back to step i) if $H$ and play $S$ forever if $T$.

iii) If there ever was a deviation from these strategies go back to step i).

Following these strategies, for player 1, the obtained value ($v^1$) will be,

$$v^1 = (1 - \delta)1 + \delta[xv^1 + (1 - x)4]$$

where $x$ is the probability of head ($H$) in the coin.

This expression basically means that, given player 2 plays M, if player 1 decides to play U instead of D (grabing only 1 and not the potential 2 by
cheating her), he is rewarded by flipping a coin that gives a chance to moving
to a best situation (specifically the NE). Otherwise, if player 1 decides to
deviate, the coin is not flipped and he got stucked in the same situation,
delaying the chance of reversion to the NE by an additional period.

For example, if we want to get $v^1 = \bar{v}^1 = 2$

$$2 = (1 - \delta)1 + \delta [x2 + (1 - x)4]$$

From here we can get the probability $x$ that makes 1 indifferent between
playing U or D such that he obtains exactly $v^1 = 2$. In this case, we need
that

$$x = \frac{3\delta - 1}{2\delta}$$

It’s important to see also that in this case $x$ is well defined for $\delta \geq \frac{1}{3}$, as
obtained for $\bar{v}^1$

It’s easy to check this is an equilibrium.

If player 2 plays M, player 1 will be indifferent between playing U or D (by construction of $x$), not having an incentive to deviate. If the coin
decides player 2 to play S, then the best response by player 1 is to play D
(this is the NE).

If player 1 plays U, player 2’s best response is effectively to play M. If
the coin decides player 1 to play D, then player 2’s best response is to play
S (again, the NE).

**Graphical Summary of LR player’s payoffs**

<table>
<thead>
<tr>
<th>Payoffs for LR player</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\min u^1$</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>